$\mathcal{C}$ symmetric monoidal category. $\mathcal{D}$ $\mathcal{C}$-enriched category

operad over $\mathcal{C}$: $\mathrm{Fin} \longrightarrow \mathcal{C}$ (sym) resp. $\mathrm{Ord} \longrightarrow \mathcal{C}$ (ns) $\Big\}$ $\mathrm{Hom}_{\mathrm{Fin}}(n;n) = S_n$, equipped with $\gamma: \mathcal{O} \circ \mathcal{O} \longrightarrow \mathcal{O}$

Let $\mathcal{O}$ be an operad. algebra over $\mathcal{O}$: $\mathcal{O} \longrightarrow \mathrm{Hom}(A, A)$ ($\in [\mathrm{Fin}, \mathcal{C}]$). $\mathrm{Hom}(A,A)(n) = \mathrm{Hom}_{\mathcal{D}}(A^{\otimes n}; A)$

Free operad. $\mathrm{Operad} \underset{T\text{- free}}{\overset{\text{forgetful}}{\rightleftarrows}} [\mathrm{Fin}, \mathcal{C}]$ $T(X) = \{$ trees with vertices assigned elements of $X \}$.

Quadratic data: $E \in [\mathrm{Fin}, \mathcal{C}]$. $R \subseteq T(E)^{(2)}$.

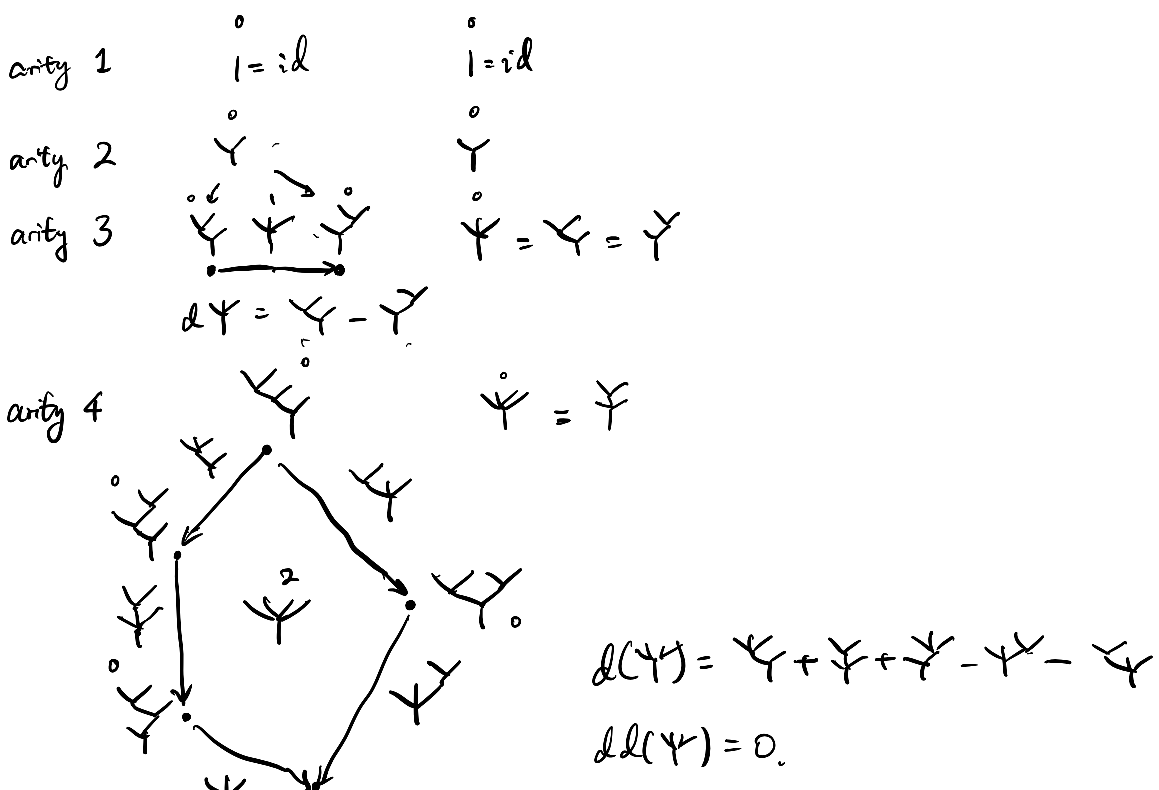
quadratic operad $\mathcal{P} = \mathcal{P}(E; R) = T(E)/R$.

$As = \mathcal{P}(Y; \; Y\!\!\!\!\!\!\text{(left comb)} - Y\!\!\!\!\!\!\text{(right comb)})$ ~~$(12)\cdot Y - Y$~~

Sym: Com $= \mathcal{P}(Y ; \ldots)$, Lie $= \mathcal{P}(Y; \ldots)$, $(12)Y = -Y$ $\quad R =$ sub-$\underline{S_3}$-module of $\underline{T(E)(3)}$

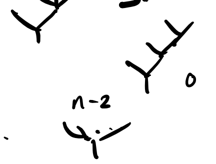
As quasi-free resolution: A_∞ ($\mathcal{C}$ = (differentially) graded $\mathbb{K}$-vector space)

$A_\infty := (T(\ldots\ldots), d) \longrightarrow As$ (graded operad concentrated on index 0)



$d(\text{tree}) = \ldots$

$dd(\text{tree}) = 0.$



arity n ...

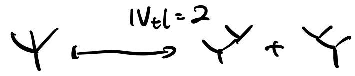
$A_\infty = (T(\text{trees of arity } 0, 1, 2, \dots), d) \xrightarrow{\text{quasi-iso.}} As$

Koszul duality of operads. given <u>quadratic</u> operad $\mathcal{P} = \mathcal{P}(E;R)$ — also work with quadratic-linear-constant $R \subseteq T(E)^{(2)} \oplus E \oplus I$.

cooperad (ns/sym) $\quad C : \mathrm{Fin} \to \mathcal{C}$ (resp. Ord)

$$\Delta : C \to C \bar{\circ} C = \bigoplus_{r \le n} (C(r) \otimes C^{\otimes r})^{S_r}(n)$$

$$\Delta_{(1)} : C \to \bigoplus_{\substack{t \in PT^n \\ |V_t| = 2}} t(C)$$



linear dual: C cooperad. $\quad P = C^*$ operad. $\quad P(n) = \mathrm{Hom}_{\mathbb{K}}(C(n), \mathbb{K})$

$$\gamma_{(i)} : P(m) \otimes P(n) \to P(m+n-1)$$

$$f \otimes g \mapsto (\underset{C(m+n-1)}{x} \mapsto (f \otimes g)(\Delta_{(i)}(x)))$$

$$(f \otimes g)(a \otimes b) = f(a) g(b), \quad a \in C(m),\ b \in C(n)$$

$\mathcal{P}$ operad $\quad$ <u>$C = P^*$ cooperad</u> ? $\quad$ if each $P(n)$ finite dimensional (ns. $\mathbb{K}$-vector spaces / sym. $\mathbb{K}S_n$-mod)

cofree (connected) cooperad:

$$T^c(X) \to X, \quad X \in [\mathrm{Fin}, \mathcal{C}]$$
$$\exists! \ C \dashrightarrow T^c(X) \text{ (tree representation)}, \quad C \to X$$
C (connected cooperad)

conilpotent LV 5.7.6.

free operad.

$$X \to T(X), \quad X \in [\mathrm{Fin}, \mathcal{C}]$$
$$X \to \mathcal{O} \text{ (operad)}, \quad \exists! \ T(X) \dashrightarrow \mathcal{O}$$

quadratic data:

$$C(E;R) \to T^c(E) \to T^c(E)/R$$
$$\exists! \ C \dashrightarrow C(E;R), \quad C \to T^c(E), \quad C \dashrightarrow T^c(E)/R \text{ is } 0$$

$C(E;R)^{(1)} = E.$

$C(E;R)^{(2)} = R$

$\vdots$

quadratic data $E; R \subseteq T(E)^{(2)}$

$$R \hookrightarrow T(E) \to \mathcal{P}(E;R) = T(E)/R$$
$$T(E) \to \mathcal{O}, \quad R \dashrightarrow \mathcal{O} \text{ is } 0, \quad \exists! \ \mathcal{P}(E;R) \dashrightarrow \mathcal{O}$$

$\mathcal{P}(E;R)^{(1)} = E$

$\mathcal{P}(E;R)^{(2)} = T(E)^{(2)}/R$

$\vdots$

Assume $\underline{s : \mathcal{C} \to \mathcal{C}}$ is the shift functor (suspension) $\underline{A \mapsto A[1]}$. (assume $\mathcal{C}$ = (diff.) graded $\mathbb{K}$-vect space)

$F \in [\mathrm{Fin}, \mathcal{C}]$. $\underline{sF(n) = s(F(n))}$

<u>Koszul dual cooperad</u> $\quad \mathcal{P}^{¡} = C(sE; s^2R)$ (anti-shriek)

operadic co-Bar construction. $\mathcal{C} = I \oplus \bar{\mathcal{C}}$ (coaugmented) $\Longrightarrow$ dg. operad ?

$I(1) = \mathbb{K}\,id$, $I(n) = 0$ for $n > 1$

Ω : coaug. coop. $\longrightarrow$ dg. operad.

$$\Omega \mathcal{C} = (\mathcal{T}(s^{-1}\bar{\mathcal{C}}), d_2) \quad \text{(LV. 6.5.5)}$$

$$d_2 : s^{-1}x \xrightarrow{\Delta_{(1)}} s^{-1}(\sum a_i \otimes b_i) \longmapsto \sum s^{-1}a_i \otimes s^{-1}b_i \xrightarrow{\gamma} \sum \cdots$$

LV 6.5.6 : $d_2^2 = 0$.

operadic twisting morphisms. dg. cooperad $\mathcal{C}$ - dg. operad $\mathcal{P}$.

$$\underline{Hom(\mathcal{C}, \mathcal{P})}(n) = Hom_{\mathbb{K}}(\mathcal{C}(n), \mathcal{P}(n)) \quad \text{(resp. } \mathbb{K}S_n\text{)}$$

$\sigma \in S_n$. $(\sigma f)(x) = \sigma(f(\sigma^{-1}x))$

$f : \{f_i \in Hom(\mathcal{C}, \mathcal{P})(i), i \in \mathbb{N}\}$

$$df = d_{\mathcal{P}} \circ f - (-1)^{|f|} f \circ d_{\mathcal{C}}$$

$$f * g := \mathcal{C}(n) \xrightarrow{\Delta_{(1)}} \bigoplus (\mathcal{C}(m) \otimes \mathcal{C}(r)) \xrightarrow{f \otimes g} \bigoplus (\mathcal{P}(m) \otimes \mathcal{P}(r)) \xrightarrow{\gamma_{(1)}} \mathcal{P}(n)$$

Maurer-Cartan equation $df + f * f = 0$.

$\hookrightarrow$ f is <u>twisting morphism</u> $\mathcal{C} \to \mathcal{P}$ if it satisfies MC

Universal prop. for Ω:

quadratic operad $\mathcal{P} = \mathcal{P}(E; R)$ $\quad \mathcal{P}^{i} = \mathcal{C}(sE; s^2R)$

natural twisting morphism $\mathcal{P}^{i} \to sE \xrightarrow{s^{-1}} E \to \mathcal{P}$.

$\hookrightarrow$ $\Omega \mathcal{P}^{i} \to \mathcal{P}$. We say $\mathcal{P}$ is Koszul if this is quasi-isomorphism.

then let $\mathcal{P}_\infty = \Omega \mathcal{P}^{i}$

test for Koszulity: rewrite method, etc. (LV chap. 8)

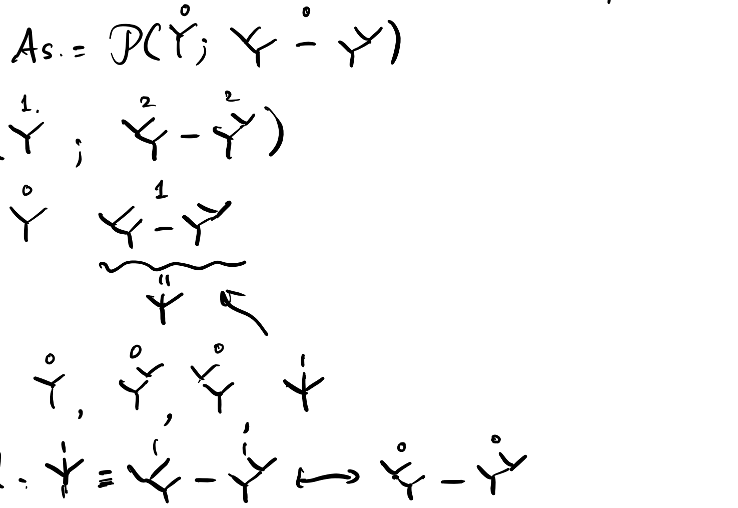
eg. Let $\mathcal{P} = As := \mathcal{P}(\ldots)$

$\mathcal{P}^{i} = \mathcal{C}(\ldots)$

$s^{-1}\bar{\mathcal{P}}^{i} \ni$

$\mathcal{T}(s^{-1}\bar{\mathcal{P}}^{i})$:

d :



$\Omega P^{i} = (T(s^{-1}\bar{P}^{i}), d)$: $\mathring{Y}, \dot{Y}, \cdots$ $\quad dY = Y - Y$

Koszul dual operad $\quad P^{!} = (S^{c} \underset{H}{\otimes} P^{i})^{*} \leftarrow$

operadic suspension / Hadamard product. $\quad Hom_{\mathbb{K}}(n) = Hom(\mathbb{K}^{\otimes n}, \mathbb{K})$

$S = Hom_{s\mathbb{K}} \qquad S(1) : Hom(s\mathbb{K}, s\mathbb{K})$ degree 0

$S(2) : Hom(s^{2}\mathbb{K}^{\otimes 2}, s\mathbb{K})$ degree -1

$\vdots$

$(S \underset{H}{\otimes} P)(n) = S(n) \otimes P(n)$

$S^{c} = Hom^{c}_{s\mathbb{K}} \qquad Hom^{c}_{s\mathbb{K}}(n) = Hom(\mathbb{K}, \mathbb{K}^{\otimes n})$

If each $E(n)$ finite dimensional, then

$\Rightarrow P^{!} = \mathcal{P}(s^{-1}S^{-1} \underset{H}{\otimes} E^{*}, R^{\perp})$, where $S^{-1} = Hom_{s^{-1}\mathbb{K}}$

sketch: $P^{i} = \mathcal{C}(sE, s^{2}R)$

then $s^{2}R \subseteq T(sE)^{(2)}$

$(s^{2}R)^{\perp} \subseteq (T(sE)^{(2)})^{*} \cong T(s^{-1}E^{*})^{(2)}$

$\Downarrow$

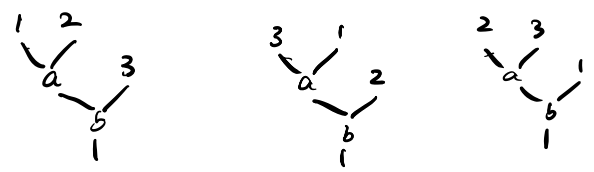
$R^{\perp} \subseteq T(s^{-1}S^{-1} \otimes E^{*})^{(2)}$

(LV, chap 8.)

When $E(n) = 0$ for $n \neq 2$ $\quad T(E)^{(2)} = T(E)(3)$

For sym: $T(E)(3) = E \otimes Ind_{S_2}^{S_3} E \cong 3\, E \otimes E$

$a \circ_{I} b. \qquad a \circ_{II} b \qquad a \circ_{III} b.$



Let $E^{\vee} = E^{*} \otimes sgn_{S_2}$ $\quad$ ($sgn_{S_2} = \mathbb{K}$ with S_2-action $(12) = -1$)

$\langle -, - \rangle$ given by $\langle a' \circ_u b', a \circ_v b \rangle = \begin{cases} a'(a)\, b'(b) & \text{if } u = v \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow P^{!} = \mathcal{P}(E^{\vee}, R^{\perp})$ where $R^{\perp} \subseteq T(E^{*})(3)$ given $\langle -, - \rangle$.

e.g. $Com^{!} = Lie$. $\quad Lie^{!} = Com$.

$(12)\,Y = Y \quad (12)\,Y = -Y$

For ns: (if $\mathcal{P}$ ns operad, then $(\mathcal{P}(n) \otimes \mathbb{K}S_n)_{n\in\mathbb{N}}$ sym operad)

$$P^! = \mathcal{P}(E^*, R^\perp)$$

$$\mathcal{T}(E)(3) = E \otimes E \oplus E \otimes E.$$

$R^\perp$ w.r.t. $\langle -, - \rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

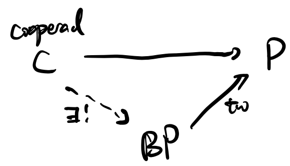
↳ e.g. $As^! = As$

<u>Sneak peek of operadic homological algebra:</u> $\mathcal{P}$ operad, what does $\mathcal{P}_\infty$-algebra look like?

$$Hom(\Omega P^i, Hom_A) = \underline{Hom(P^i, B\,Hom_A)} \cong Tw(P^i, End_A) = Codiff(P^i(A)).$$

(Bar construction: $\mathcal{P}$ operad (augmented: $P = I \oplus \bar{P}$)

then $BP = (T^c(s\bar{P}), d_2)$ is dg. cooperad



$\Omega \dashv B$

$f \in Tw(P^i, End_A)$. $f_n \in Hom(P^i(n), End(A^{\otimes n}, A)) \cong Hom(P^i(n) \otimes A^{\otimes n}, A)$

$P^i(n) \otimes A^{\otimes n} =: P^i(A)(n)$ $\quad$ ($P^i(A)$ is a P^i-coalgebra)

then $Hom(P^i, End_A) \cong Coder(P^i(A))$

$\quad\quad \cup \quad\quad\quad\quad \cup$

$Tw(P^i, End_A) \cong Codiff(P^i(A))$ ← square-zero elements.

A, B are P_∞-algebra. then ∞-morphism $A \to B$. is a morphism $P^i(A) \to P^i(B)$

$P^i(A) \to P^i(B)$ uniquely determined by $\underline{P^i(A) \to B}$ satisfying some axioms.

$\quad\quad\quad$ ‖ by tensor-hom

$\underline{P^i \to End_B^A}$ $\quad End_B^A(n) = Hom(A^{\otimes n}, B)$

$\uparrow$

$P = As.$ $\quad P^i \ni Y, Y, Y \cdots$

Homotopy Transfer Theorem: if $\mathcal{P}$ is Koszul operad, (H, d_H) is homotopy retract of (A, d_A).

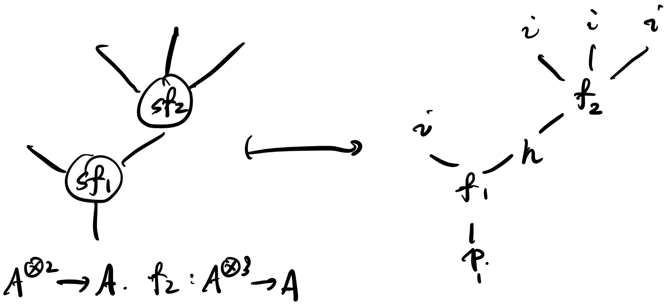
$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

$Id_A - ip = d_A h + h d_A.$ $\quad$ i: quasi-iso then any P_∞-structure on A can be transferred to H s.t. i becomes ∞-quasi-isom.

Proof: we have element in $\mathrm{Hom}(P^{i}, B\mathrm{Hom}_A)$, we want element in $\mathrm{Hom}(P^{i}, B\mathrm{Hom}_H)$

$\Psi : B\mathrm{Hom}_A \to B\mathrm{Hom}_H$.

$T^c(s\mathrm{Hom}_A) \to \mathrm{Hom}_H$



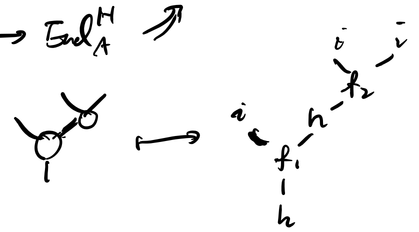
$f_1 : A^{\otimes 2} \to A$. $f_2 : A^{\otimes 3} \to A$

given $v: P^{i} \to \mathrm{Hom}_A$

explicitly $H^{\otimes 2} \to H$, $H^{\otimes 3} \to H$ … ? equivalently $P^{i} \to \mathrm{Hom}_H$

given by $P^{i} \xrightarrow{\Delta} T^c(P^{i}) \xrightarrow{T^c(sv)} T^c(s\mathrm{Hom}_A) \to \mathrm{Hom}_H$

$i_\infty : P^{i} \to \mathrm{End}_A^H$



when $H = H(A)$ $\quad A_n = B_n \oplus H_n(A) \oplus B_{n-1}$

i_∞. $H(A) \to A$. $\quad p_\infty : A \to H(A)$.

$\Rightarrow$ given A, B P_∞-algebra, with quasi-isom. $f: A \to B$.

$\tilde{f} : H(A) \xrightarrow{p_\infty} A \xrightarrow{f} B \xrightarrow{i_\infty} H(B)$ – co-isomorphism.

$f^{-1} : B \xrightarrow{i_\infty} H(B) \xrightarrow{\tilde{f}^{-1}} H(A) \xrightarrow{p_\infty} A$.

$\therefore$ all P_∞-quasi-isomorphisms are invertible.